Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x=\dfrac{\sqrt{3}(z+1)}{z-1}$, $z^3-1=u^3$.

$$\therefore \int\frac{dx}{(x+\sqrt{3})\sqrt[3]{(x^2+1)}}=-\int\frac{dz}{z\sqrt[3]{[4(z^3-1)}}=-\int\frac{udu}{\sqrt[3]{4(u^3+1)}}.$$

$$\therefore\ y=\frac{1}{3\sqrt[3]{4}}\int\frac{du}{1+u}-\frac{1}{6\sqrt[3]{4}}\int\frac{(2u-1)du}{1-u+u^2}-\frac{1}{2\sqrt[3]{4}}\int\frac{du}{1-u+u^2}.$$

$$\therefore\ y=\frac{1}{6\sqrt[3]{4}}\log\left(\frac{(1+u)^2}{1-u+u^2}\right)-\frac{1}{\sqrt{3}\sqrt[3]{4}}\tan^{-1}\left(\frac{2u-1}{\sqrt{3}}\right),$$

where $u=\dfrac{\sqrt[3]{[6\sqrt{3}(x^2+1)]}}{x-\sqrt{3}}$.

170. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find the center-locus of conics having 4-point contact with a given conic at a given point. Show that the conic of minimum eccentricity is given by $e^4\tan^2\varphi+4e^2-4=0$, where e is its eccentricity, and φ is the angle which the linear center-locus above makes with the normal to the curve at the point.

Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics in the State University, Athens, Ohio.

The coördinates of the center of any conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \ldots\ldots (1),\text{ are}$$

$$x_1=\frac{hf-bg}{ab-h^2}\ldots\ldots(2),\ \ y_1=\frac{gh-af}{ab-h^2}\ldots\ldots(3),$$

and the eccentricity is given by

$$e^4+\frac{(a-b)^2+4h^2}{ab-h^2}(e^2-1)=0\ldots\ldots(4).$$

If the tangent and normal to the given curve be the axes of abscissas and ordinates, the equation of the conic having 4-point contact with the given conic is of the form

$$ax^2+2hxy+by^2+2gx-\lambda x^2=0\ldots\ldots(5),\text{ or, }(a-\lambda)x^2+2hxy+by^2+2gx=0\ldots\ldots(6).$$

Comparing (1) and (6), $a=a-\lambda$, $f=0$, $c=0 \ldots\ldots (7)$, and (2) and (3) become

$$x_1=-\frac{bg}{(a-\lambda)b-h^2}\ldots\ldots(8),\quad y_1=\frac{gh}{(a-\lambda)b-h^2}\ldots\ldots(9).$$

(9)÷(8) gives $y_1-(h/b)x_1 \ldots\ldots (10)$, the locus-center. Thus the "slope" is

$-(h/b)$, and the tangent of the angle (10) makes with the y axis is $\tan\varphi=-(b/h)$.....(11). Substituting $a=a-\lambda$ in (4), we find

$$u=\frac{1}{e^2}=\frac{a-\lambda+b}{\sqrt{[(a-\lambda-b)^2+4h^2}}+\tfrac{1}{2}.....(12).$$

Equating $du/d\lambda$ to zero, we find $(a-\lambda)b-b^2-2h^2=0$, or,

$$a-\lambda=\frac{b^2+2h^2}{b}.....(13).\quad \text{Then } a-\lambda-b=\frac{2h^2}{b}.....(14).$$

Substituting (7), (13) and (14) in (4) and reducing

$$e^4+\frac{4h^2}{b^2}(e^2-1)=0, \text{ or } \frac{b^2}{h^2}e^4+4e^2-4=0.....(15).$$

This becomes by (11), $e^4\tan^2\varphi+4e^2-4=0$.....(16).

Also solved by *G. W. GREENWOOD*, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill., and by *G. B. M. ZERR*, A. M., Ph. D., Parsons, W. Va.

171. Proposed by J. E. SANDERS, Hackney, Ohio.

A thread passes spirally around a *rough* cylinder 10 feet high and 6 inches in diameter. How far will a pigeon fly in unwinding the thread if the distance between the coils is 4 inches, and the thread *unwound* is at all times *horizontal?*

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let r=radius of cylinder=3 inches=$\frac{1}{4}$ feet; d=distance between coils =$\frac{1}{3}$ feet; m=number of coils=30. Then from Vol. I, No. 6, pages 318-320 of this Journal* we have for the required distance

$$S=\pi m^2\sqrt{(4\pi^2r^2+d^2)} \text{ feet}=900\pi\sqrt{(\tfrac{1}{4}\pi^2+\tfrac{1}{9})} \text{ feet}=4540.246 \text{ feet, nearly.}$$

II. Solution by H. B. LEONARD, B. S.

Circumference=6π. Length of thread on one turn=$\sqrt{[(6\pi^2)+4^2]}$. Number of turns=$(12\times10)\div4=30$. Total rotation=60π. Angle of elevation of thread on cylinder=$\alpha=\sin^{-1}\dfrac{4}{\sqrt{[(6\pi^2)+4^2]}}$. During the bird's flight, an unwinding of $d\theta$ produces an increase of $dz=3\tan\alpha.d\theta$ in altitude, of $dr=3\sec\alpha.d\theta$ in direction of flight, of $dc=3\sqrt{(\theta^2\sec^2\alpha+1)}d\theta$ normal to direction of flight.

$$(ds)^2=(dr)^2+(dz)^2+(dc)^2=(9\sec^2\alpha+9\tan^2\alpha+9\theta^2\sec^2\alpha+9)(d\theta)^2$$
$$=(18\sec^2\alpha+9\theta^2\sec^2\alpha)(d\theta)^2.\quad ds=3\sec\alpha\sqrt{(2+\theta^2)}d\theta.$$

$$S=\int_0^{60\pi}3\sec\alpha\sqrt{(2+\theta^2)}d\theta=3\sec\alpha\int_0^{60\pi}\sqrt{(2+\theta^2)}d\theta=3\sec\alpha\Big[\tfrac{1}{2}\{\theta\sqrt{(2+\theta^2)}$$

* See also Vol. 1, pp. 88-89. Ed.